ON THE TWO-DIMENSIONAL MOTION OF A THICK ELLIPTICAL WING UNDER A FREE SURFACE

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PMM Vol.26, No.4, 1962, pp. 797-800

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(Received February 2, 1962)

The problem of the steady irrotational motion of a wing under a free surface will be considered in curvilinear coordinates ρ, v, connected by a conformal mapping

$$z = \omega(\zeta), \qquad \zeta = \rho e^{iv} \tag{0.1}$$

of a circular annulus onto the region bounded by the contour of the wing and the x-axis. The circles ρ = const in the ζ-plane correspond to the curves ρ = const in the z-plane; we let the contour of the wing correspond to the unit circle, and the x-axis to the circle $\rho = \rho_2$. We also require that the point at infinity in the z-plane correspond to a point on the line $v = 0$ in the ζ-plane.

1. We assume that the wing moves along the positive x-direction with velocity c. We limit ourselves to the case of small values of the parameter $\lambda = 2gh/c^2$, i.e. the case of large Froude numbers [1,2].

Since Laplace's equation in the curvilinear coordinates ρ, v, has the same form as in polar coordinates, the velocity potential can be given in the form of a series

$$\varphi = \frac{\Gamma}{2\pi} v + \sum_{m=1}^{\infty} (A_m \rho^m + A_{-m} \rho^{-m}) \cos mv + \sum_{m=1}^{\infty} (B_m \rho^m + B_{-m} \rho^{-m}) \sin mv \tag{1.1}$$

Here Γ is the circulation around the wing, taken positive in the positive direction of v

The constants $A_i B_i$ may be found from the boundary conditions on the free surface [1] and on the contour of the wing

$$\varphi = 0 \quad \text{for} \quad \rho = \rho_2, \qquad \frac{\partial \varphi}{\partial n} = c \cos(n, x) \quad \text{for} \quad \rho = 1 \tag{1.2}$$

The condition of impermeability may be written as [3]

$$\frac{\partial \varphi}{\partial \rho} = c\,\frac{\partial y}{\partial v} = \sum_{m=1}^{\infty} (\alpha_m^{(1)} \cos mv + \beta_m^{(1)} \sin mv) \quad \text{for } \rho = 1 \tag{1.3}$$

The parameters $\alpha_i^{(1)}$, $\beta_i^{(1)}$, are determined from the mapping function (0.1).

In order to satisfy the condition on the free surface, we replace the first term in (1.1) by a Fourier series

$$v = 2\sum_{1}^{\infty} \frac{(-1)^{m-1}}{m} \sin mv \qquad (-\pi < v < \pi) \tag{1.4}$$

Substituting (1.1) into the boundary condition (1.2), and comparing the coefficients of the same trigonometric functions, we find A_m, A_{-m}, B_m, B_{-m}.

For the velocity potential, we get

$$\varphi = \frac{\Gamma\vartheta}{2\pi} + \frac{\Gamma}{\pi}\sum_{1}^{\infty}(-1)^m \frac{\rho_2^{\,m}}{m(1+\rho_2^{\,2m})}(\rho^m + \rho^{-m}) \sin mv +$$

$$+\sum_{1}^{\infty} \frac{\alpha_m^{(1)}}{m(1+\rho_2^{\,2m})}(\rho^m - \rho^{2m}\rho_2^{\,-m}) \cos mv + \sum_{1}^{\infty} \frac{\beta_m^{(1)}}{m(1+\rho_2^{\,2m})}(\rho^m - \rho_2^{\,2m}\rho^{-m}) \sin mv \tag{1.5}$$

We show that the velocities, corresponding to the chosen potential, vanish at infinity, i.e. $V_x^{\infty} = V_y^{\infty} = 0$. We know that the pole of the function $\omega(\zeta)$ determines the point at infinity of the inverse function, thus

$$\frac{d\zeta}{dz} = \frac{1}{\omega'(\zeta)} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{for } |z| \to \infty$$

From (1.4) and (1.5), it is clear that for $v = 0$ and $1 \leqslant \rho \leqslant \rho_2$, the derivatives $\partial\varphi/\partial\rho$ and $\partial\varphi/\partial v$ are bounded in absolute value. Consequently,

$$V_x^{\infty} = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial x} = 0, \qquad V_y^{\infty} = \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{1.6}$$

Knowing $\varphi(\rho, v)$, we find the conjugate stream function $\psi(\rho, v)$ and the expression for the complex potential

$$w(\zeta) = \varphi(\rho, v) + i\psi(\rho, v) = \frac{\Gamma}{2\pi i}\ln \zeta + \sum_{1}^{\infty}(b_m\zeta^m + b_{-m}\zeta^{-m}) + iD \tag{1.7}$$

$$b_m = A_m - iB_m, \qquad b_{-m} = A_{-m} + iB_{-m}$$

The constant of integration D is determined from the condition

$$\psi = 0 \quad \text{for } x = \pm\infty \qquad \text{or} \quad \psi(\rho_2, 0) = 0$$

2. To clarify the proposed method of solution, let us consider the motion of a wing of almost elliptical shape at large depths.

The function, giving the conformal mapping of a circular annulus onto the given region, has the form

$$z = \omega(\zeta) = \frac{iB}{1 - \varkappa\zeta} + A - \frac{iB}{2} + (Q + iP)\sum_{1}^{\infty} \varkappa^{3m}\zeta^{m} + (Q - iP)\sum_{1}^{\infty} \varkappa^{m}\zeta^{-m} \qquad (2.1)$$

where A, B, Q, P, κ are real parameters, to be determined, in which $0 < \kappa < 1$.

Formula (1,2) is so chosen that $\mathrm{Im}[\omega(\zeta)] = y = 0$ for $\rho = \kappa^{-1}$. Thus $\rho_2 = \kappa^{-1}$.

On the contour of the wing $\zeta = \sigma = e^{iv}$, the mapping function may be represented as a Laurent series ($|\kappa\sigma| < 1$

$$\omega(\sigma) = iB\sum_{1}^{\infty} \varkappa^{m}\sigma^{m} + A + \frac{iB}{2} + (Q + iP)\sum_{1}^{\infty} \varkappa^{3m}\sigma^{m} + (Q - iP)\sum_{1}^{\infty} \varkappa^{m}\sigma^{-m} \qquad (2.2)$$

or

$$\omega(\sigma) = \frac{iB}{1 - \varkappa\sigma} - \frac{iB}{2} + A + (Q + iP)\frac{\varkappa^3\sigma}{1 - \varkappa^3\sigma} + (Q - iP)\frac{\varkappa}{\sigma - \varkappa} \qquad (2.3)$$

To calculate A, B, C and κ, we assume that, approximately

$$\omega(\sigma) \approx \omega_1(\sigma) = A - \frac{iB}{2} + \frac{iB}{1 - \varkappa\sigma} + (Q - iP)\frac{\varkappa}{\sigma - \varkappa} =$$

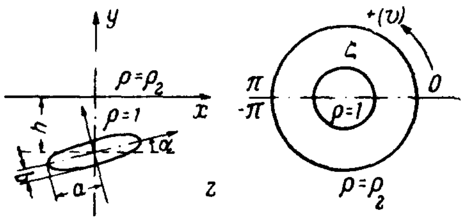
$$= A + \frac{iB}{2} + iB\sum_{1}^{\infty} \varkappa^{m}\sigma^{m} + (Q - iP)\sum_{1}^{\infty} \varkappa^{m}\sigma^{-m} \qquad (2.4)$$

The justification of this assumption will be carried out later. Isolating in (2.4) the real and imaginary parts, we obtain the approximate equations for the wing contour in parametric form

$$x \approx x_1 = -(B + P)S_2 + QS_1 + A$$

$$y \approx y_1 = (B - P)S_1 - QS_2 + \frac{B}{2} \qquad (2.5)$$

$$S_1 = \sum_{1}^{\infty} \varkappa^{m}\cos mv, \qquad S_2 = \sum_{1}^{\infty} \varkappa^{m}\sin mv$$



We show that (2.5) is the equation of an ellipse in parametric form.

We first note that

$$\left(S_1 - \frac{\varkappa^2}{1-\varkappa^2}\right)^2 + S_2^2 = \left(\frac{\varkappa}{1-\varkappa^2}\right)^2 \quad \text{for } |\varkappa| < 1 \tag{2.6}$$

This is easily seen, if we consider the auxiliary function

$$z_0 = x_0 + iy_0 = \frac{\varkappa\sigma}{1-\varkappa\sigma} = S_1 + iS_2 \quad \text{for } |\varkappa\sigma| < 1 \tag{2.7}$$

It is known that the geometrical locus of the point (x_0, y_0) is a circle, whose equation coincides with (2.6). Solving the system (2.5) for S_1 and S_2, and substituting the resulting expressions into (2.6), we obtain the equation of the ellipse

$$\left[\frac{(x_1-A)\,Q-(y_1-B/2)\,(B+P)}{Q^2+P^2-B^2} - \frac{\varkappa^2}{1-\varkappa^2}\right]^2 + \left[\frac{(x_1-A)(B-P)-(y_1-B/2)\,Q}{Q^2+P^2-B^2}\right]^2 = \left(\frac{\varkappa}{1-\varkappa^2}\right)^2 \tag{2.8}$$

Considering the ellipse to have an angle α with respect to the x-axis, and its center to have the coordinates $(0, -h)$, we find

$$A = -\frac{Q\varkappa^2}{1-\varkappa^2}, \qquad \frac{B}{2} - \frac{\varkappa^2(P-B)}{1-\varkappa^2} + h = 0, \qquad \tan 2\alpha = \frac{Q}{P} \tag{2.9}$$

For the equation of the ellipse in canonical form $x^2/a^2 + y^2/b^2 = 1$, we have

$$\frac{\varkappa^2}{(1-\varkappa^2)^2}\,\frac{(Q^2+P^2-B^2)^2}{Q^2+P^2+B^2-2\sqrt{Q^2B^2+P^2B^2}} = a^2 \tag{2.10}$$

$$\frac{\varkappa^2}{(1-\varkappa^2)^2}\,\frac{(Q^2+P^2-B^2)^2}{Q^2+P^2+B^2+2\sqrt{Q^2B^2+P^2B^2}} = b^2 \tag{2.11}$$

B, P, Q and κ are found from (2.9), (2.10) and (2.11).

The signs of B, P and Q must be so chosen that $0 < \kappa < 1$ holds. For elongated profiles ($a/b \gg 5$) and small $\alpha(|\alpha| < 15^\circ)$

$$B = -\frac{1-\varkappa^2}{2\varkappa}(a+b) = -2h + \varkappa\,[a+b-(a-b)\cos 2\alpha] \tag{2.12}$$

$$P = -\frac{1-\varkappa^2}{2\varkappa}(a-b)\cos 2\alpha, \qquad Q = -\frac{1-\varkappa^2}{2\varkappa}(a-b)\sin 2\alpha \tag{2.13}$$

$$\varkappa = \frac{-2h + \sqrt{4h^2 + 2\,(a^2-b^2)\cos 2\alpha - (a+b)^2}}{2\,(a-b)\cos 2\alpha - (a+b)} \tag{2.14}$$

Here h is the distance of the center of the ellipse from the free surface, a and b are the semi-axes (Fig. 1). In the following table are calculated the parameter κ for some values of $h/2a$, a/b and α.

	$\alpha=0$			$\alpha=15°$		
$h/2a$	1.0	1.5	2.0	1.0	1.5	2.0
$a/b=5$	0.153	0.105	0.075	0.162	0.108	0.098
$a/b=10$	0.136	0.089	0.071	0.138	0.091	0.076
$a/b=15$	0.132	0.087	0.069	0.133	0.089	0.073

From (2.14), we see that as h increases, the parameter κ decreases. Substituting A, B, P, Q and κ in Expression (2.1), we find the desired mapping function $\omega(\zeta)$.

In conclusion, we verify the correctness of the assumption (2.4). From (2.3) and (2.4) we have

$$\omega(\sigma) = \omega_1(\sigma) + \Delta\omega(\sigma), \qquad \Delta\omega(\sigma) = (Q + iP)\frac{\varkappa^3\sigma}{1-\varkappa^3\sigma} \tag{2.15}$$

Here $z_1 = \omega_1(\sigma)$ is the equation of the ellipse in complex form.

We estimate the absolute value of $\Delta\omega(\sigma)$; considering (2.13) and (2.15), we obtain the inequality

$$|\Delta\omega(\sigma)| = \frac{\varkappa^2(1-\varkappa^2)}{2\sqrt{1-2\varkappa^3\cos v + \varkappa^6}}(a-b) < \frac{\varkappa^2}{2}(a-b)$$

For $h/2a \geqslant 1$ and $5 \leqslant a/b \leqslant 15$, we have $|\Delta\omega(\sigma)| < 0.12b$ according to the table and (2.14), i.e. the contour $x + iy = \omega(\sigma)$ is close to an elliptic contour $x_1 + iy_1 = \omega_1(\sigma)$.

3. We consider the problem of the motion of an almost elliptic wing at large depths. Separating in (2.2) the real and imaginary parts, we find

$$y(1, v) = \frac{B}{2} + (B - P)\sum_1^\infty \varkappa^m \cos mv + P\sum_1^\infty \varkappa^{3m}\cos mv - Q\sum_1^\infty \varkappa^m(1-\varkappa^{2m})\sin mv \tag{3.1}$$

From Formula (1.3) we determine the parameters $\alpha_m^{(1)}$ and $\beta_m^{(1)}$

$$\alpha_m^{(1)} = cmQ\varkappa^m(\varkappa^{2m}-1), \qquad \beta_m^{(1)} = cm\varkappa^m(P - B - P\varkappa^{2m}) \tag{3.2}$$

Substituting (3.2) into (1.5), and remembering that $\rho_2 = \kappa^{-1}$, we get

$$\varphi = \frac{\Gamma v}{2\pi} + cQ\sum_1^\infty \frac{\varkappa^m(\varkappa^{2m}-1)}{\varkappa^{2m}+1}(\varkappa^{2m}\rho^m - \rho^{-m})\cos mv + \tag{3.3}$$

$$+\frac{\Gamma}{\pi}\sum_1^\infty \frac{(-1)^m\varkappa^m(\rho^m+\rho^{-m})}{m(1+\varkappa^{2m})}\sin mv + c\sum_1^\infty \frac{\varkappa^m(P-B-P\varkappa^{2m})}{1+\varkappa^{2m}}(\varkappa^{2m}\rho^m - \rho^{-m})\sin mv$$

Determining $\varphi(\rho, v)$, we write the complex potential as (1.7).

In a similar manner, we may solve the problem of a wing moving near a rigid wall.

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Translated by C.K.C.